

# LAPLACE EQUATIONS FOR THE DIRAC, EULER OPERATORS AND THE HARMONIC OSCILLATOR

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**ABSTRACT:** In this article, we give the explicit solutions to the Laplace equations associated to the Dirac operator, Euler operator and Harmonic oscillator in  $\mathbb{R}$ .

**Key words:** Dirac operator, Euler operator, Harmonic oscillator, Laplace equation.

## 1 – Introduction:

In the paper [4] we have solved the wave Cauchy problems associated to the Dirac, Euler operator and the harmonic oscillator.

The aim of this paper is to give the explicit solutions to the following Cauchy problems of Laplace type

$$\begin{cases} \left(D + \frac{\partial^2}{\partial y^2}\right) u(y, X) = 0 & ; (y, X) \in \mathbb{R}_+^* \times \mathbb{R} \\ u(0, X) = u_0(X) & ; u_0 \in C_0^\infty(\mathbb{R}) \end{cases} \quad (LD)$$

$$\begin{cases} \left(E_a + \frac{\partial^2}{\partial y^2}\right) v(y, \xi) = 0 & ; (y, \xi) \in \mathbb{R}_+^* \times \mathbb{R} \\ v(0, \xi) = v_0(\xi) & ; v_0 \in C_0^\infty(\mathbb{R}) \end{cases} \quad (LE_a)$$

$$\begin{cases} \left(H^a + \frac{\partial^2}{\partial y^2}\right) w(y, x) = 0; (y, x) \in \mathbb{R}_+^* \times \mathbb{R} \\ w(0, x) = w_0(x) & ; w_0 \in C_0^\infty(\mathbb{R}) \end{cases} \quad (LH_a)$$

where:

$$D = \frac{\partial}{\partial X}, \quad (1.1)$$

$$E^a = -2a\xi \frac{\partial}{\partial \xi}, \quad a > 0 \quad (1.2)$$

$$H^a = \frac{\partial^2}{\partial x^2} - a^2 x^2, \quad a > 0 \quad (1.3)$$

are respectively the Dirac operator, the Euler operator and the harmonic oscillator on  $\mathbb{R}$ . These operators play a fundamental role in many mathematical and physical problems. In physics the harmonic oscillator appears

when e.g. modeling atoms and their quantum states ([2], [3]). Note that the Laplace equation associated to the Euler operator ( $LD$ ) is a typical example of a hyperbolic equation with multiple characteristics considered by Leray in 1960 (see[1], P.355).

## 2– Laplace equation for the Dirac and the Euler operators

Here our objective is to solve the Cauchy problems ( $LD$ ) and ( $LE_a$ ), for this we need the following recall:

Let  $\Omega \subset \mathbb{R}$  be an open set and let  $\nu \geq 1$  be a fixed real number.

The class  $G^\nu(\Omega)$  of Gevrey functions of order  $\nu$  in  $\Omega$  is the set of functions  $f \in C^\infty(\Omega)$  satisfying the property that for every compact subset  $K$  of  $\Omega$ , there exists a positive constant  $C = C_K$  such that for all  $l \in \mathbb{N}$  and all  $x \in K$

$$|\partial^l f(x)| \leq C^{l+1} (l!)^\nu.$$

It is easy to recognize that  $G^1 = A(\Omega)$ , the space of all analytic functions in  $\Omega$ .

Assume  $\nu > 1$ , we shall denote by  $G_0^\nu(\Omega)$  the vector space of all  $f \in G^\nu(\Omega)$  with compact support in  $\Omega$  (see[5]).

**Theorem 2.1:** The Cauchy problem ( $LD$ ) for the Laplace equation associated to the Dirac operator has the unique solution given by:

$$U(y, X) = \int_X^{+\infty} P_D(y, X, X') U_0(X') dX' \quad (2.1)$$

where

$$P_D(y, X, X') = \frac{y}{2\sqrt{\pi}} (X' - X)^{-3/2} \exp \left\{ \frac{-y^2}{4(X' - X)} \right\} \quad (2.2)$$

**Proof.** Note that the Cauchy Problem for the Laplace equation associated to the Dirac operator is well-posed in  $G^\nu(\mathbb{R})$  for any  $1 < \nu < 2$  see [7].

The fact that the function  $P_D(y, X, X')$  satisfies the Laplace equation can be checked by direct computation. To see the limit condition we use the change of variables  $s = \frac{y^2}{4(X' - X)}$ .

**Corollary 2.2:** The Cauchy problem ( $LE_a$ ) for the Laplace equation associated to the Euler operator has the unique solution given by:

$$V(y, \xi) = \int_{|\xi'| < |\xi|} P_{E^a}(y, \xi, \xi') V_0(\xi') d\xi' \quad (2.3)$$

where  $L_{E^a}(y, \xi, \xi')$  is given by

$$P_{E^a}(y, \xi, \xi') = \sqrt{\frac{a}{2\pi}} \frac{y}{\xi'} \log^{-3/2}(|\xi/\xi'|) \exp \left\{ \frac{-ay^2}{2\log(|\xi/\xi'|)} \right\} \quad (2.4)$$

**Proof:** This can be shown by using the change of variable  $X = \frac{\log|\xi|}{-2a}$  and the theorem 2.1.

### 3-Laplace equation for the Harmonic oscillator

**Theorem 3.1** The Cauchy problem  $(LH^a)$  for the Laplace equation associated to the Harmonic oscillator has the unique solution given by:

$$w(y, x) = \int_{-\infty}^{+\infty} P_{H^a}(y, x, x') w_0(x') dx' \quad (3.1)$$

where  $P_{H^a}(y, x, x')$  is given by

$$P_{H^a}(y, x, x') = \frac{\sqrt{a}y}{2\pi} \int_0^{+\infty} u^{-3/2} \cdot (\sinh(2au))^{-1/2} \times \\ \exp \left\{ \frac{-y^2}{4u} - \frac{a}{2} (x^2 + x'^2) \coth(2au) + \frac{axx'}{\sinh(2au)} \right\} du \quad (3.2)$$

**Proof:** We recall the integral kernel of the heat operator for the harmonic oscillator (Mehler-Fuchs- formula, [2])

$$K_a(t, x, x') = \sqrt{\frac{a}{2\pi}} \frac{1}{\sqrt{\sinh(2at)}} \times \\ \exp \left[ -\frac{a}{2} (x^2 + x'^2) \coth(2at) + \frac{axx'}{\sinh(2at)} \right] \quad (3.3)$$

Taking the derivative of the formula (6], p.50)

$$\frac{e^{-t\lambda}}{t} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-ut^2} u^{-1/2} e^{-\lambda^2/4u} du \quad (3.4)$$

with respect to  $\lambda$  we can write

$$\frac{e^{-t\lambda}}{t} = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-ut^2} u^{-3/2} \lambda e^{-\lambda^2/4u} du \quad (3.5)$$

By setting  $t = \sqrt{-H_a}$  and  $\lambda = y$  in (3.5) we get

$$e^{-y\sqrt{-H_a}} = \frac{y}{\sqrt{2\pi}} \int_0^\infty e^{-s^2/4u} u^{-3/2} e^{uH_a} du \quad (3.6)$$

where  $e^{uH_a}$  is the heat operator of the harmonic oscillator. Let  $P_{H_a}(y, x, x')$  be the integral kernel of  $e^{-y\sqrt{-H_a}}$ , making use of the formula (3.3) in (3.6) we see that the formula (3.2) holds and the proof of the theorem 3.1 is finished.

**Proposition 3.1** Let  $P(y, x, x') = \frac{1}{\pi} \frac{y}{y^2 + (x-x')^2}$  be the Poisson kernel on the half- plane ([6], p.60). Then we have

$$\lim_{a \searrow 0} P_{H_a}(y, x, x') = P(y, x, x'). \quad (3.7)$$

**Proof:** The formula (3.7) uses essentially the formula (3.2) and  $\lim_{s \searrow 0} \frac{s}{\sinh(s)} = 1$ .

### 4-Directions for further studies:

We suggest here a certain number of open related problems connected to this paper. We are interested in the Laplace equation for the harmonic oscillator with an inverse square potential

$$\begin{cases} \frac{\partial^2}{\partial y^2} u(y, x) + \left( \frac{\partial^2}{\partial x^2} - a^2 x^2 - \frac{b^2}{x^2} \right) u(y, x) & (y, x) \in \mathbb{R}_+^* \times \mathbb{R} \\ u(0, x) = f(x) & f \in C_0^\infty(\mathbb{R}) \end{cases}$$

Finally, we suggest problems in direction of the non linear Laplace equations for the harmonic oscillator and to look for global solution and a possible blow up in finite times.

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